ACTEX Academic Series

Mathematics & Models for Financial Derivatives

First Edition

1st

Chris Wilson, PhD, ASA

Chris Wilson

Mathematics and Models for Financial Derivatives

ACTEX Learning · Greenland, New Hampshire

Request for permission should be addressed to:
ACTEX Learning
PO Box 69
Greenland, NH 03840
support@actexmadriver.com

Manufactured in the United States of America

Cover Design by Rachel Maragnano

ISBN: 978-1-64756-778-1

Table of Contents

to Donna

who

in every aspect of life

and

in every way

never fails to inspire

Preface

When I first taught financial derivatives at Butler University over a decade ago, I struggled to find a text that was a good match for the academic backgrounds of my students, nearly all of whom were actuarial science majors. I wanted a text that would support them in achieving several specific objectives:

- Develop an intuitive understanding of how derivative options and transactions can be used strategically to solve problems or accomplish financial goals, especially at the level represented by official actuarial examinations.
- Become fluent in the mathematics of how payoff and pricing models work.
- Experience the mathematical aspects of the subject with efficiency and a level of rigor appropriate to an undergraduate actuarial science, mathematics, or quantitative finance student.
- Use theory and lots of concrete examples side-by-side as "twin pillars", so that application and theoretical understanding are mutually reinforced.
- Engage with the material via a concise, readable (yet precise), and friendly resource.

There was a gap to bridge between the best available MBA-style texts (which are perfectly appropriate for an audience of MBA students) and meeting the objectives I've listed above. I created many resources for my students *to tailor an approach to their specific needs and background as actuarial science, mathematics, and quantitative finance students.* This text, *Mathematics and Models for Financial Derivatives*, brings together that tailored approach in a single, easy-to-use resource.

Whether you are an undergraduate student, an actuary, or a faculty member, you will find this to be a user-friendly text. If you've not previously encountered the material, you will soon see that the mathematical content is satisfying and diverse in flavor, with a wide variety of interesting applications. The writing style intentionally brings out relationships between calculus and financial concepts in a way that strengthens command of the material.

An additional goal I had for this text was **maximum flexibility.** Here's what I mean:

- **Minimal prerequisites.** The text assumes only a little calculus and some basic probability (conditional probability, normal random variables). In fact, many students have been completely successful using this textbook, even without first taking a formal course in interest theory or probability.
- **Accommodates small programs.** Because courses often must be offered on alternating years in smaller programs, being able to offer a course with minimal prerequisites makes it possible for the greatest number of students to study financial derivatives (successfully!) whenever it is offered.
- **Specifically written for the actuarial science, mathematics, or quantitative finance undergraduate.** We provide a readable, friendly, well-organized resource with logical progression of topics, especially of the mathematics. The writing and development is specifically geared to make this special group of students feel "at home" in a great math class.
- **Flexibility for the new instructor.** If you've never taught or learned about financial derivatives before, this book is for you! All of the terms are clearly defined, and all of the math is written out and organized (and justified!) in a way that makes sense and will feel familiar to you. If you can graph a piecewise-linear function and know a bit of calculus and probability, you are ready to enjoy teaching from this book with confidence!
- **Flexibility for the seasoned veteran.** Have you taught the old Exam MFE/3F or IFM syllabi? Do you wish you had an efficiently laid-out resource but don't want to change all of your notation? Nearly all of the notation is identical to what's been used on actuarial exams for years—you'll find yourself at home in this text. You'll notice lots of options (see below) to accommodate the way you like to present the material.

As of the publication date, all of the financial derivatives syllabus topics for Society of Actuaries Exam FAM are covered in detail within Chapters 1, 2, 3, 5, and Section 8.3. A number of former actuarial exam questions and sample problems have been incorporated as exercises; these are marked with (*).

The heart of the text is the first five chapters. The first chapter introduces financial derivatives (forwards, futures, puts, calls), terminology, applications, and several basic formulas and identities. From there the text takes up the question of what puts and calls should cost, and binomial pricing models are the subject of Chapter 2. Two choices are provided for introducing binomial models. The default option (if you read things in order) first demonstrates how to construct and use the multiperiod "finished product" to help the reader develop familiarity with the context, and then a replicating portfolio argument justifies the computations. An alternate starting point is offered in the Chapter 2 Appendix, beginning with the problem of selling a call and hedging by purchasing "just the right amount" of the underlying asset for a single binomial time interval.

By considering properties that one would want to retain in a continuous limiting case of the binomial model, we arrive at the lognormal asset price model (Chapter 3). Then the *100% optional* Chapter 4 covers stochastic differential equations and Itô's Lemma, culminating in an accessible derivation of the Black-Scholes Partial Differential Equation. (With an eye towards maximum flexibility, none of the Chapter 4 material is prerequisite to the remainder of the text.) Chapter 5 derives the Black-Scholes option pricing formulas and demonstrates hedging using option Greeks. Instructors may choose to include anything from minimal to complete coverage of theory for any of these topics and will find that they are well-supported by the text. It really is meant to be *as flexible as possible.*

Chapters 6 through 8 can be thought of as a big menu of additional applications and techniques to be used as desired: Monte Carlo simulation methods, interest rate models/derivatives, real options, corporate valuation applications, and annuity guarantees.

Our default context (as is the case throughout most financial derivatives literature) is to consider a share of stock $S(t)$ paying continuous dividends to its owner. Many other variations and contexts are covered (discrete dividends, commodities, currencies, compound options, to name a few), and these variations may be included or omitted at the discretion of the reader or instructor.

Acknowledgments. I would like to express my sincere thanks and gratitude to everyone at Actex and Archimedia Advantage for their diligent efforts to help bring this text to completion: Kim Neuffer at Actex helped me so much (Thank you!) with keeping this project on track and getting it to publication. Yijia Liu and his team of LaTeX wizards did nearly all of the coding and diagrams. Bill Morella kindly showed a lot of flexibility in supporting this project at various stages, for which I am grateful. I'd also like to mention my appreciation for Stephen Camilli who was so helpful to me in the early writing stages and Garrett Doherty who produced some of the graphs. A number of reviewers and proofreaders provided insightful feedback, in particular Caryn Chuah, Ong Li Kee, Gary Dean, Mary Krohn, Jenny Cox, and Karen Holmes all contributed in very significant ways. I also appreciate the many thought-provoking phone calls I had with Michael Bean from the comfort of my extremely makeshift 2020 home office and our conversations about financial derivatives. Finally, the author appreciates the availability of Society of Actuaries sample exam problems (in this text these are marked with (*) in the problem sets) and the SOA's continued practice of allowing these to be incorporated into textbooks used by aspiring actuaries.

About the author. Chris Wilson, PhD, ASA, is a Professor of Actuarial Science at Butler University, Indianapolis, where he has been a faculty member at since 2007. He has led the actuarial science program for over a decade and was chair of what is now called the Department of Mathematical Sciences from 2018-2021. His publications include articles about the pedagogy of actuarial science as well as articles in noncommutative ring theory. He is a classically trained pianist and enjoys making music with others and traveling with his family.

1

The Basics of Financial Derivatives

Imagine that you can purchase a contract that will give you the right to purchase a share of Google stock one year from now for \$2800. Such a contract could prove to be very valuable if the stock performs well during the year. On the other hand, if the price of one share turns out to be \$2750 at the end of the year, then the contract will expire with no value.

Now imagine another contract that gives the owner the right to sell a share of Microsoft at any time during the next six months for a selling price of \$300. This contract could be viewed as a sort of insurance policy that could protect the owner of a share of Microsoft against near-term economic losses if the stock's value drops. On the other hand, the contract pays the holder nothing if the stock is performing well.

The contracts described here are examples of **financial derivatives**. The term "derivative" indicates that the value of the contract is in some way dependent upon (or *derived from*) the price of another asset (in our illustration, a share of Google or Microsoft). A right to purchase an asset is referred to as a **call**, and a right to sell is called a **put**.

Several interesting questions come to mind. What should it cost to purchase the contracts described above? What is such a contract worth one month after its purchase if the stock has performed well? If the stock has not performed well? How can portfolios of contracts such as these be used as tools to manage risks? Are there ways in which these contracts themselves are risky?

In this text, we will study standard mathematical models that address these questions. In this first chapter of the text, we will learn about how we can assemble the most basic financial derivatives (forward contracts, calls, and puts) into portfolios that can accomplish various risk management goals.

1.1 Time Value of Money

An essential concept to understanding financial derivative pricing models is something called **time value of money**. Suppose I want to have \$5000 in an interest-bearing account at the end of the year. This can be accomplished in a variety of ways. I could wait for the whole year, and deposit $X = \$5000$ at the end. Or instead, I could do some calculations and figure out an amount Y to deposit now so that it will grow to \$5000 over the year. A third option would be to wait one month, and deposit an amount Z. There's a sense in which X, Y, and Z are equivalent: \$5000 at time $t = 1$ (we'll always measure time in years) can be achieved via Y at time $t = 0$ or via Z at time $t = 1/12$. Indeed, there is a whole spectrum of monetary values at various points in time that are each equivalent to the time $t = 1$ value \$5000. As we will see, there are factors (called accumulation factors and discount factors) by which we can multiply to convert from \$5000 at time $t = 1$ to the equivalent amount at any other time t.

1.1.1 Linear Approximations and Differential Equations

Many of the financial economics concepts that we will consider in this text, including that of time-value-of-money, can be boiled down to understanding how small changes in value accumulate over time. Let us briefly walk through the underlying calculus concepts.

You are probably familiar with the derivative of a function f at a point:

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \tag{1.1}$$

This gives the rate at which some quantity f changes with respect to small changes in the input x_0. We could reformulate this as

$$f'(x_0) = \frac{f(x_0 + h) - f(x_0)}{h} + (\text{small error term}), \tag{1.2}$$

where the small error term approaches 0 quickly as h becomes small (assuming f is differentiable at x_0). If we discard the error term but keep h small, we get the approximation

$$f'(x_0) \approx \frac{f(x_0 + h) - f(x_0)}{h}, \tag{1.3}$$

which can be rearranged as follows:

$$f(x_0 + h) - f(x_0) \approx f'(x_0) \cdot h \tag{1.4}$$

That is to say, the change in the output of f is approximated by a linear function[1] of the change in input.

We say that (1.4) gives the **linear approximation** at $x = x_0$ to the change in f. As long as f is differentiable, the error in this approximation can be made as small as we like if we stick to small enough h's. We express this at the infinitesimal level by writing

$$df = f'(x)\ dx, \tag{1.5}$$

which says in symbols that the amount of change in f is approximately given by the product of the instantaneous rate $f'(x)$ with the change in input.

Equation (1.5) is said to be a **differential equation** (the df and dx factors are called **differentials**). As we will see throughout the text, differential equations are useful in understanding how small changes in a function's value accumulate over the course of many short time intervals. We illustrate this point by examining time-value-of-money under a continuously compounded interest rate.

1.1.2 The Continuously Compounded Risk-Free Rate

Deposit some money into a savings account (or into a money market account, or purchase a risk-free bond[2]). Let $X(t)$ denote the balance of the account (or market value of the bond) at time t. (We always measure time in years unless stated otherwise.) Let dt be the width of a very brief time interval.

Assuming interest rates hold steady, we would expect the amount by which the balance increases during a short stretch of time to be proportional to the amount that is in the fund at that time, so that the interest credited during any time interval $[t, t+dt]$ is some fixed percentage of the time-t accumulated balance $X(t)$. It would make sense for this percentage to be roughly proportional to the duration dt of the time interval (a larger dt should result in a larger amount of interest being paid) as long as dt is very small. So the percentage by which $X(t)$ will increase in dt years can be expressed as $r \cdot dt$ for some annualized rate r. We express this mathematically with the differential equation

$$dX(t) = r \cdot X(t)\ dt, \tag{1.6}$$

[1] Linear in the "linear algebra sense"—view $f'(x)$ as a matrix and h as an "input vector."

[2] Unless otherwise specified, we mean a zero-coupon bond; that is, a bond that does not make intermediate interest payments (which are sometimes called coupon payments) prior to maturity.

which says that during the time interval $[t, t+dt]$, the fund will increase by approximately $r \cdot X(t) \cdot dt$ dollars, and this approximation is good for small enough dt. The factor r in this expression is called the **continuously compounded risk-free rate of interest** and is always quoted on an annualized basis. A country's risk-free rate is the rate earned when one invests in that country's debt via a government bond or treasury security. We will assume throughout the text that $r > 0$.

Example 1.1. An investor makes a single deposit into an account that earns the continuously compounded risk-free rate. Let $X(t)$ denote the time-t balance of the account. If the account grows according to the differential equation

$$dX(t) = .04X(t)\ dt \tag{1.7}$$

(a) Identify the continuously compounded risk-free rate.

(b) Approximate the amount of interest earned during the week that follows time $t = 2$ if the time $t = 2$ balance $X(2)$ is \$12,000.

Solution.

(a) Direct comparison with (1.6) reveals that the risk-free rate is $r = .04$ or 4% per year.

(b) One week is a fairly short stretch of time. Take $dt = 1/52$ in Equation (1.7) to get the approximation

$$dX(2) \approx .04\ \underbrace{X(2)}_{12{,}000}\ \underbrace{dt}_{1/52} \approx \$9.23. \tag{1.8}$$

We can interpret this computation in the following way: A week is 1/52 of a year, so we take 1/52 of the risk-free rate 4% and increase the 12,000 balance by 1/52 of 4%. In general, if we view the "dt" in Equation (1.6) to be a small fraction of the year, then $r \cdot dt$ gives the *percentage* by which the account value increases during that time period.

$\square$

Our differential equation (1.6) describes increases in the account value over short stretches of time. We can use an integral to add these small changes together over longer stretches of time in an effort to express the time-t account balance $X(t)$ in terms of r and t.

It will be helpful to rearrange (1.6) a little bit, collecting factors that contain $X(t)$ on the left side of the equation:

$$\frac{1}{X(t)} dX(t) = r\ dt \tag{1.9}$$

Now, add these small amounts of change over the time interval $[0, T]$ using an integral. (We're taking the limit as $dt \to 0$ of the sum of the small changes represented in Equation (1.9).)

$$\int_{t=0}^{T} \frac{1}{X(t)} dX(t) = \int_{t=0}^{T} r\ dt, \tag{1.10}$$

$$\ln(X(t))\Big|_{t=0}^{T} = rt\Big|_{t=0}^{T} \tag{1.11}$$

Using logarithmic properties, we have

$$\ln\left(\frac{X(T)}{X(0)}\right) = rT \tag{1.12}$$

$$\frac{X(T)}{X(0)} = e^{rT}. \tag{1.13}$$

Multiplying by $X(0)$, we obtain the following theorem:

Theorem 1.2 (Accumulated Value at the Continuously Compounded Risk-Free Rate)**.**

(a) If $dX(t) = r \cdot X(t)\ dt$ for some constant r, then $X(T) = X(0)e^{rT}$ for all T.

(b) Let r denote the continuously compounded risk-free interest rate. A deposit of $X(0)$ into a risk-free investment will grow to a value at time T of

$$X(T) = X(0)e^{rT}. \tag{1.14}$$

Theorem 1.2 says that we can multiply by an **accumulation factor** of the form e^{rT} to find the equivalent value of a transaction at a later date or by a **discount factor** of the form e^{-rT} to find an equivalent value of a transaction at some date prior to the transaction date.

Example 1.3. You will receive a payment of \$1000 in exactly nine months (that is, at time $t = .75$). You could opt instead to receive an equivalent amount of money (i) right now, (ii) at time $t = .5$, or (iii) at time $t = 1.25$. Find these amounts if the continuously compounded risk-free rate is 4%.

Solution.

(i) Solving $X(.75) = X(0)e^{r\times.75}$ for $X(0)$, we get

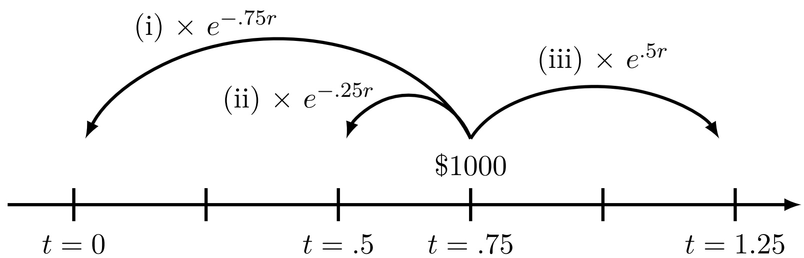
$$X(0) = X(.75)e^{-.75r} = 1000e^{-.75(.04)} \approx \$970.45.$$

(ii) This is similar, but we are finding an equivalent dollar amount for a transaction occurring .25 years prior to the time $t = .75$ payment. So we can multiply the \$1000 time $t = .75$ payment by the factor $e^{-.25r}$:

$$X(.5) = 1000e^{-.25r} = 1000e^{-.25\times.04} \approx \$990.05$$

(iii) To what amount will \$1000 grow after a half-year time period? Theorem 1.2 implies that we may multiply the \$1000 by $e^{.5r}$ to find the amount:

$$X(1.25) = 1000e^{.5r} = 1000e^{.5\times.04} \approx \$1020.20$$



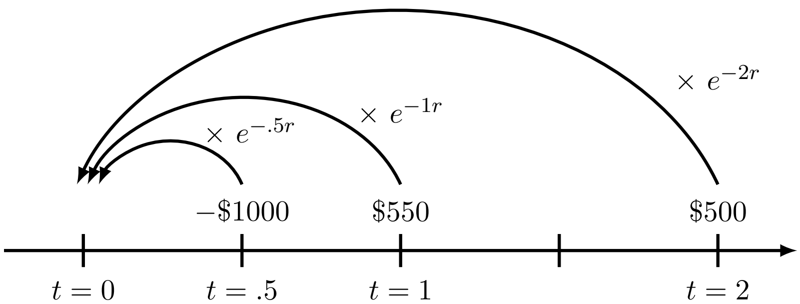
□

Example 1.4. You have an arrangement in which you will spend \$1000 at time $t = .5$. You will receive \$550 at time $t = 1$ and \$500 at time $t = 2$. Find (i) the present value (i.e. the time $t = 0$ value) and (ii) the time $t = 3$ value of this set of transactions if $r = 4\%$.

Solution.

(i) The time $t = 0$ value of a set of transactions is equal to the sum of the time $t = 0$ values of the individual transactions. The time $t = 0$ value is

$$-1000e^{-.04(.5)} + 550e^{-.04(1)} + 500e^{-.04(2)} \approx \$9.79.$$



A quick way to enter this on a calculator is to store the "one-year discount factor" $e^{.04}$ by entering $e^{.04}$ [STO] [X,t,θ,n] on a TI graphing calculator (on the TI-30XS MultiView use the [$xyztabc$] key instead of [X,t,θ,n]). Then the time $t = 0$ present value can easily be found by entering the polynomial $-1000X^{1/2}+550X+500X^2$ into the calculator. (The discount factor "X" in this expression is commonly denoted by v.)

(ii) Time $t = 3$ is 2.5 years later than the \$1000 expenditure, 2 years later than the \$550 income, and 1 year later than the \$500 income. Thus the time $t = 3$ value of these transactions is

$$-1000e^{.04(2.5)} + 550e^{.04(2)} + 500e^{.04(1)} \approx \$11.04.$$

$\square$

1.1.3 Interest Compounded at an Annual Effective Rate

Another common way to quote interest rates is the use of **annual effective rates**, which name the percentage by which an amount on deposit grows in one year's time. Understanding the mathematics of annual effective rates is quite simple: To increase a quantity by, say, 6%, you can multiply that quantity by 1.06 (i.e. find 106% of the original amount). To increase the *resulting* quantity by 6%, you could simply multiply by a second factor of 1.06 to obtain (original amount) $\times(1.06)^2$. If the annual effective interest rate is $i = .06$, then a deposit of X will grow to $X \cdot (1.06)^2$ at time $t = 2$. We have the following result:

Theorem 1.5 (Accumulated Value at an Annual Effective Interest Rate)**.** Let i denote the annual effective rate of interest. Then a deposit of $X(0)$ into a risk-free investment will grow to an amount $X(T)$ at time T equal to

$$X(T) = X(0)(1+i)^T. \tag{1.15}$$

Proof. The explanation given in the previous paragraph justifies the statement for integer values of T. In particular, we have

$$X(1) = X(0) \cdot (1+i). \tag{1.16}$$

If an account balance grows at a steady percentage rate throughout the year, then the differential equation $dX(t) = r \cdot X(t)\, dt$ must hold for some constant rate r. By Theorem 1.2, we have

$$X(T) = X(0)e^{rT} \tag{1.17}$$

for all $t \geq 0$. Equations (1.16) and (1.17) give us two ways to write $X(1)$; equating these, we get $X(0) \cdot (1+i) = X(1) = X(0)e^{r \cdot 1}$. Thus

$$1 + i = e^r. \tag{1.18}$$

Substituting into Equation (1.17), we get

$$X(T) = X(0)(1+i)^T \tag{1.19}$$

for all $t \geq 0$. □

A commonly-used notation for the one-year discount factor is the symbol v. So we have

$$v = (1+i)^{-1} = e^{-r}. \tag{1.20}$$

Theorem 1.5 says that we can multiply by an accumulation factor of the form $(1+i)^T$ to find the equivalent value of a transaction at a later date or multiply by a discount factor of the form $v^T = (1+i)^{-T}$ to find an equivalent value at a previous date.

Example 1.6. Let us rework Example 1.4 in the context of effective annual interest rates: You will spend \$1000 at time $t = .5$ in order to receive \$550 at time $t = 1$ and \$500 at time $t = 2$. Find (i) the present value (i.e. the time $t = 0$ value) and (ii) the time $t = 3$ value of this set of transactions if $i = 3.5\%$.

Solution.

(i) The time $t = 0$ value of these transactions is

$$-1000(1.035)^{-.5} + 550(1.035)^{-1} + 500(1.035)^{-2},$$

which can be written as

$$-1000v^{.5} + 550v^{1} + 500v^{2}$$

where $v = (1.035)^{-1}$. As in Example 1.4, the use of the v notation suggests a quick method of calculator entry: Store the value $(1.035)^{-1}$ as X in the calculator and then enter the polynomial

$$-1000X^{.5} + 550X^{1} + 500X^{2}.$$

This gives a present value of approximately \$15.21.

(ii) The time $t = 3$ value is

$$-1000(1.035)^{2.5} + 550(1.035)^{2} + 500(1.035) \approx \$16.86.$$

□

1.2 Forward Contracts

Consider the following scenarios:

- An investor anticipates that a significant amount of capital will become available three months from now. The investor would like at that time to invest in a particular stock and would like to lock in a price that is equivalent (in a sense which will we will soon describe) to the price at which the stock is currently trading. By locking in a purchase price now, the investor does not have to worry that the desired stock will become prohibitively expensive when the capital needed to purchase the stock becomes available.

- A large candy manufacturing company knows that it will need a large quantity of cocoa to support its holiday production schedule in six months. The market price is currently favorable, and by locking in a purchase price based on the current market, the manufacturer can arrange to have the cocoa delivered when it is needed (rather than storing it) and at the same time eliminate the risk of a significant increase in the price of cocoa. Such an arrangement benefits the cocoa producer as well, because sudden drops in the price of cocoa would not affect the contracted purchase price.

- A corporation will make a sizeable transaction in a few weeks involving a foreign currency. By locking in today's exchange rate as the rate to be used when the future transaction occurs, both parties are protected against an outcome in which that party's home currency experiences a sizeable loss of value compared to the other currency.

In each scenario, a contract is described in which it is agreed that an asset—be it a stock, a commodity, or a foreign currency—will be purchased (or, for the counterparty in the contract, the asset will be sold) in the future using a contractual price that is based on today's market price. These contracts are known as **forward contracts** or, more briefly, simply as **forwards**.

As these examples illustrate, forward contracts remove risk and uncertainty from future transactions—the party who does the buying is protected from increases in the asset price, and the party who does the selling is protected from decreases in the asset price. Unlike some of the other contract types introduced later in this chapter, both parties are *obligated* under a forward contract to complete the purchase/sale as described by the contract's terms. The date on which the transaction occurs is called the **delivery date** and a forward contract's agreed-upon asset price is said to be the contract's **forward price** or **delivery price**.

Forward contracts are examples of **derivative** instruments, the term "derivative" indicating that the contract itself has value that is *derived from* another asset (e.g. the stock, the cocoa, or the foreign currency). The particular asset involved in a financial derivative contract is known as the **underlying asset**, or the **underlying** for short.

As we hinted above, if the underlying asset is not delivered until some future point in time, and if the agreed-upon purchase price is not paid until that future point in time, there are a couple of factors that come into play in determining a "suitably equivalent" price to today's **spot price**, that is, the price for which the asset can be purchased today in the marketplace. Consider, for example, a forward contract in which the underlying asset is a stock and will be purchased three months from now for a price of $\$F$.

- A dollar paid three months from now is not equivalent in the sense of time-value-of-money to a dollar right now.
- If the stock pays dividends during the three-month life of the forward contract, those dividends will not be received by the recipient of the stock when the contract is settled in three months—they should be removed from the price in an appropriate way.

Because the forward price is determined in a manner that is equivalent to the spot price with respect to these two issues, the contract itself does not have an initial cost to either party or any value at the time the contract is written and signed. (As the stock price changes over time, the contract may gain or lose value, as we will see.)

Assumptions

One typically makes a number of common simplifying assumptions when studying financial derivatives and their values and cash flows. We will make the following assumptions throughout this text:

- We assume that the continuously compounded risk-free rate (denoted throughout the text by r) is constant.

  We are actually assuming two things here. First, this assumes that the prices and maturity values of government treasury securities reflect a single yield rate r, regardless of the amount of time until a particular treasury security reaches its maturity date. (One says that the yield curve is flat when this is the case.)

  Second, we are assuming that the rate r does not change over time in the sense that, after a year (for example) passes, the rate r implied by bond prices will be the same as the rate r implied by today's bond prices.

In reality, neither of these simplifying assumptions usually holds, but for the time intervals typically involved in real-life derivative contracts, it is not an unreasonable assumption.

- We assume that investment assets can always be traded (either bought or sold) by the investor as needed—assume that the necessary counterparty to buy or sell shares of an asset is always available.
- We assume that any fraction of a share of an asset such as a stock can be bought or sold by the investor as needed.
- We assume that the investor can borrow or lend at the risk-free rate whenever necessary.

Some of these assumptions are clearly never true (the yield curve for government bonds is not always "flat", as assumed here) or may be less true for the smaller-scale investor than for the large firm (for example, the ability to borrow at/near the risk-free rate). These are, nevertheless, useful assumptions and are implicitly present in many of the mathematical models and formulas that are used to price and determine the value of financial derivative contracts.

No-arbitrage assumption. We will also make frequent use of an important principle in financial economics called the **no-arbitrage assumption**. ("Arbitrage" is pronounced in a way that rhymes somewhat with how a Midwesterner from the U.S. might say "car garage".) The term **arbitrage** refers to a risk-free way to earn a profit. The no-arbitrage assumption states that arbitrage opportunities do not occur in an efficient marketplace—we will make frequent use of this assumption.

The no-arbitrage assumption implies a principle that is sometimes known as the **Law of One Price**: If you can find two methods to construct a particular time-T payoff, then the two methods of arranging that payoff must have the same cost (that is, equivalent costs in the sense of time-value-of-money). If "Portfolio 1" and "Portfolio 2" had different costs but the same time-T payoff, then one could finance a huge purchase of the lower priced portfolio by selling the higher priced portfolio, pocketing the difference at time T when payments and asset delivery have been settled. Arbitrageurs would act quickly on such mispriced portfolios. The resulting frequent trading would drive the prices of the portfolios toward each other, quickly eliminating the opportunity for risk-free profit. As we now turn our attention to the determination of the delivery price for a forward contract, the Law of One Price will justify our computations.

1.2.1 Pricing and timing of delivery and payment (continuous and zero dividend cases)

Suppose you would like to be in possession of a stock at a future time T. You could take possession of the stock either now at time 0 or later at time T. In either case, you might pay for the stock now at time 0 or later at time T.

You could decide to enter a contract (now, at time 0) in which you pay for the stock now but take possession later at a contractually agreed-upon price. Such an agreement is called a **prepaid forward** contract. What amount should you pay now to receive the stock at time T?

You could alternatively enter (at time 0) a contract that has both the timing of delivery of the stock and of the required payment of the agreed-upon purchase price both occurring in the future at time T. This is the **forward** contract arrangement described at the start of this chapter. What amount should you agree at time 0 to pay at time T under the forward contract?

Notation and terminology. Let us introduce some permanent notation that will allow us to discuss forward and prepaid forward contracts. For all times $t \geq 0$, let $S(t)$ denote the price at time t (t is always measured in years) of the stock or other asset involved in the contract. As above, denote the continuously compounded risk-free rate by r. Note that, although a stock's **spot price** $S(0)$ is known, $S(t)$ is not knowable in advance for $t > 0$. (Later chapters in this text will demonstrate two common and highly useful ways to model $S(t)$ as a random variable for $t > 0$.)

Suppose our stock or asset pays dividends at a continuous rate δ. Like r, this rate is quoted on an annualized basis (regardless of whether "per year" is explicitly stated). The phrase "paying dividends at a continuous rate δ" means that, during every short time interval $[t,\ t+dt]$, the owner of the stock receives a dividend payment of $\delta \cdot S(t) \cdot dt$ from the corporation for which the stock provides a share of corporate ownership. Most of the examples and models explored in the text assume that dividends are paid continuously—we will be explicit when this is not the case. This assumption always includes stocks that do not pay dividends—set δ equal to 0 for such stocks.

Let us consider four common ways to pair the time at which an asset is acquired with the time at which the purchase price is paid. Each of these four methods will require a different amount to be paid by the buyer of the asset.

Outright purchase

At time 0, one can simply pay the spot market price $S(0)$ and take immediate possession of the asset. Especially in contexts where the other purchasing options may be under discussion, the term **outright purchase** is used to describe this transaction.

Fully leveraged purchase

A **fully leveraged purchase** is a buy-now, pay-later arrangement. The party who wishes to acquire the asset takes immediate possession (at time 0) and agrees to pay at a contractually specified future time T. This amounts to a loan of amount $S(0)$; so the party acquiring the stock will be required to pay $S(0)e^{rT}$ at time T, that is, to repay the loan along with the interest that has accrued during $[0, T]$.

Forward contract

A T-year **forward contract** is an agreement, contracted at time $t = 0$, that the purchaser of the asset will at future time T pay and receive the asset at a contractually specified price. We will soon show that the correct price to pay at time T for delivery at time T is $S(0)e^{rT} \cdot e^{-\delta T}$, or $S(0)e^{(r-\delta)T}$. We write

$$F = S(0)e^{(r-\delta)T}. \tag{1.21}$$

The price F paid at time T is referred to as the **forward price**. We will justify Equation (1.21) mathematically in Theorem 1.8. For now, let us give an informal explanation: $S(0)e^{rT}$ is the time-T amount that is equivalent (in the time-value-of-money sense) to the time-0 purchase price $S(0)$. However, this price includes the value of dividend payments that occur during $[0, T]$ which will *not* be delivered to the buyer of the asset under the forward contract. The buyer of the asset should therefore *not* pay the full amount $S(0)e^{rT}$. Theorem 1.8 will demonstrate that the correct way to remove the price of the dividend stream is to multiply by a factor of $e^{-\delta T}$.

Prepaid forward contract

A **prepaid forward contract** is an agreement, contracted at time $t = 0$, in which the party acquiring the asset pays at time $t = 0$ to receive the asset at a future time T. As was the case with forward contracts, the party receiving the asset will *not* receive the dividends paid out during $[0, T]$. Adjusting the forward contract's purchase price to its time-0 equivalent (that is, multiplying Equation (1.21) by e^{-rT}), we find that the **prepaid forward price** is

$$F^P = S(0)e^{-\delta T}. \tag{1.22}$$

The price formulas in Equations (1.21) and (1.22) reflect two basic principles to which we alluded earlier:

- Start with the correct "time-value-of-money equivalent" to the current asset price $S(0)$. That is, for contracts requiring payment at time 0, start with $S(0)$; for contracts requiring payments at time T, start with $S(0)e^{rT}$.
- Don't pay for dividends that won't be received. If the delivery of the stock (and hence, the start of the stream of dividend payments) occurs at time T, then remove from the purchase price those dividends corresponding to the time interval $[0, T]$ by multiplying by $e^{-\delta T}$.

Example 1.7. A stock is currently trading at \$50. The stock pays dividends at a continuous 2% rate. The continuously compounded risk-free rate is 5%. Give the amounts and timings involved for each of the four above-described methods of asset acquisition for an investor who would like to be in possession of this stock three months from now.

Solution.

(a) Outright purchase is simple to describe. The investor spends \$50 at time 0 and immediately receives the stock (and immediately begins to receive the dividend payments from the stock as well).

(b) Fully leveraged purchase. The investor receives the stock at time 0, either explicitly or implicitly borrowing \$50. Assuming (as we are) that the investor has the ability to borrow at the risk-free rate, the investor will pay

$$\$50e^{r\times .25} = \$50e^{.05\times .25} \approx \$50.63$$

at time $t = .25$ to pay off the loan. The investor begins receiving dividend payments immediately at time 0 under this arrangement.

(c) Prepaid forward contract. The investor pays

$$\$50e^{-\delta\times.25} = 50e^{-.02\times.25} \approx \$49.75$$

at time 0 and will receive delivery of one share of the stock at time $t = .25$. Note that the investor does not begin to receive dividends from the stock until time .25 when the stock has been delivered. Dividends paid prior to time .25 are received by the party that owned the stock at the time.

(d) Forward contract. The only thing that happens at time 0 is that the investor signs a forward contract with a counterparty. Three months later, the investor pays

$$50e^{(r-\delta)\times.25} = 50e^{(.05-.02)\times.25} \approx \$50.38$$

and receives the asset at that time. As with the prepaid forward, the investor does not receive dividends until time .25 when delivery of the asset takes place.

□

Tailed positions and the pricing formulas

We can provide a more rigorous justification and explanation for the formulas in Equations (1.21) and (1.22) by considering the following: One way to arrange (at time 0) possession at time T of one share of an asset is to enter a prepaid forward contract. An alternate method is to employ a strategy known as **tailing**, or entering a **tailed position**. This is an arrangement in which a portion of a share of the asset is purchased at time 0, and the dividend stream is reinvested as it is received to obtain more of that asset.

Theorem 1.8. Consider an investment asset, such as a stock, that pays dividends continuously at annual rate $\delta \geq 0$.

(a) If, at time 0, $N(0) = e^{-\delta T}$ shares of the asset are purchased and a tailed position is employed, then there will be $N(T) = 1$ share of the asset at time T.

(b) The correct **prepaid forward price**, paid at time 0 for delivery of the asset at time T, is $F^P = S(0)e^{-\delta T}$.

(c) The correct **forward price**, paid at time T for at time-T delivery of the asset, is $F = S(0)e^{(r-\delta)T}$.

Proof.

(a) Each share generates dividend income of $\delta\cdot S(t)\cdot dt$ during the time interval $[t, t+dt]$, so if $N(t)$ shares are held at time t, the total amount of income available for the purchase of additional shares will be $\delta\cdot S(t)\cdot N(t)\cdot dt$. At time t, each share costs $S(t)$, so reinvesting the dividend income in additional shares will increase the number of shares owned by $\delta\cdot N(t)\cdot dt$ shares. Hence, we have

$$dN(t) = \delta \cdot N(t)\, dt\,. \tag{1.23}$$

(Here we have used our assumption that transactions involving fractional numbers of shares are always possible.) In Exercise 8, you will verify that, because the number of shares owned by the investor increases incrementally according to Equation (1.23), the initial purchase $N(0) = e^{-\delta T}$ shares will grow to $N(T) = 1$ share at time T as a result of the tailing strategy.

(b) By (a), a tailed position with an initial purchase of $e^{-\delta T}$ shares has the same payoff structure as a prepaid forward contract: In both cases, the ultimate result is the possession of one share at time T with the investor not having access to any of the intermediate dividend income (in the tailed position, this income is immediately dedicated to the purchase of additional shares.) By the Law of One Price, the correct prepaid forward contract price F^P is equal to the purchase price of the $e^{-\delta T}$ shares needed for the tailed position, and that price is $S(0)e^{-\delta T}$.

(c) It is now easy to justify the formula for a forward contract's forward price F. The end result of a forward contract and the end result of a prepaid forward contract are the same: The investor has paid some money (albeit at different points in time) to acquire one share of the asset at time T. The Law of One Price implies that the amount of investment for these contracts must be equivalent in the time-value-of-money sense. The equivalent amount at time-T (i.e. the forward delivery price F) to the time-0 amount F^P is

$$F = F^P\cdot e^{rT} = S(0)e^{(r-\delta)T}.$$

$\square$

1.2.2 The value of an existing forward contract

Let us illustrate how to find the value of a forward contract that was negotiated sometime in the past.

Example 1.9. At time $t = 0$, Investor A entered a one-year forward contract ("Contract A") to acquire a dividend-paying stock.

- Under Contract A, the delivery price for the stock is \$51.
- The continuously compounded risk-free rate is $r = .05$.
- The continuously payable dividend rate for the stock is $\delta = .02$.

Nine months have passed, so today's date is represented by $t = .75$. The stock price is now \$50.90.

(a) Suppose that today (at time $t = .75$), Investor B enters a three-month forward contract (call it "Contract B") to acquire the stock. What is the delivery price for Contract B?

(b) At time $t = .75$, what is the present value of the difference between the delivery prices for Contracts A and B?

Solution.

(a) Under Contract B, the delivery price required to be paid at time $t = 1$ (which is .25 years from "today") is

$$S(.75) \cdot e^{(r-\delta)\times .25} = 50.90 \cdot e^{(.05-.02)\times .25} \approx \$51.283.$$

(b) The forward contract held by Investor A offers an advantage over Contract B! Contract A's \$50.90 delivery price, payable at time $t = 1$, is \$0.283 less than the delivery price for Contract B. The time $t = .75$ value of this difference in delivery prices is

$$.283e^{-r\times .25} = .283e^{-.05(.25)} \approx \$.2795.$$

At time $t = .75$, the value of the advantage in holding Contract A over the forward contracts that are available in the marketplace (e.g. Contract B) is evidently \$.2795. If Investor A wishes to sell her forward contract (transferring to the purchaser of the forward contract the obligation to buy one share at the lower \$50.90 delivery price at time $t = 1$), she could at time $t = .75$ sell her forward contract at a price of \$.2795. □

Example 1.9 illustrates that the value of an existing forward contract is "today's" time-value equivalent of the amount by which that contract's forward price is less than the forward price for a hypothetical "newly written today" forward contract with the same delivery date.

Formula. Let $0 < t_0 < T$, and consider a forward contract that was written at time $t = 0$ with maturity date T. The time-t_0 value of holding the forward contract is

$$\underbrace{e^{-r(T-t_0)}}_{\text{I}} \cdot \Big[\underbrace{S(t_0)e^{(r-\delta)(T-t_0)}}_{\text{II}} - \underbrace{S(0)e^{(r-\delta)T}}_{\text{III}} \Big]. \tag{1.24}$$

In words, this formula values an existing forward contract by (I) adjusting the time-value (from the delivery date to today's date) of the difference: (II) (forward price for time-T delivery if a new contract is set up today) minus (III) (forward price for time-T delivery from existing forward contract).

Note that, if the underlying stock has performed poorly, it is possible for an existing forward contract to have a negative value. This is the case when the holder of the contract has an obligation to buy the asset at a price that is higher than that required by a "new" forward contract that could be written at time t_0.

1.2.3 Forward Contracts on Assets Paying Discrete Dividends

Consider a forward contract on a stock that pays a single dividend prior to the delivery date. The intuitive principle of "Don't pay for dividends that you won't receive" from the continuous setting can be used to determine the forward price:

Theorem 1.10. Consider an asset that pays a single dividend of amount D at time t_0 where $0 < t_0 < T$. Let r denote the continuously compounded risk-free rate, and let i denote the effective annual rate. We will show that

- **(a)** The correct **prepaid forward price**, paid at time 0 for delivery of the asset at time T, is $F^P = S(0) - De^{-rt_0} = S(0) - D(1+i)^{-t_0}$.
- **(b)** The correct **forward price**, paid at time T for delivery of the asset at time T, is $F = F^P \cdot e^{rT} = S(0)e^{rT} - De^{r(T-t_0)} = S(0)(1+i)^T - D(1+i)^{T-t_0}$.
- **(c)** Multiple discrete dividends: If dividends of amounts $D_1, \ldots, D_n$ are paid at times $t_1, \ldots, t_n$ $(< T)$, then replace De^{-rt_0} in (a) by the sum $\sum D_i e^{-rt_i}$ of the present values of the dividend payments. The relationship $F = F^P \cdot e^{rT}$ then determines the forward price.

Proof.

To justify (a), we will demonstrate that a certain tailed position in the underlying asset will replicate the payoff structure of the forward/prepaid forward contracts. Suppose that, at time 0, an investor purchases $N(0) = 1 - \frac{D}{S(0)}e^{-rt_0} = 1 - \frac{D}{S(0)}(1+i)^{-t_0}$ shares of the asset and simultaneously enters a forward agreement to buy an additional $\frac{D}{S(0)}e^{-rt_0}$ shares immediately prior to the time-t_0 dividend payment. We may use Equation (1.21) with $\delta = 0$ to determine the forward delivery price for delivery immediately prior to t_0, because no dividends are paid during $[0,\, t_0)$; this price is $\frac{D}{S(0)}e^{-rt_0} \cdot S(0)e^{(r-0)t_0} = D$. The total number of shares owned—after the forward purchase has been made but before the dividend is paid—is $\left(1 - \frac{D}{S(0)}e^{-rt_0}\right) + \frac{D}{S(0)}e^{-rt_0} = 1$ share. Then at time t_0, the dividend D is paid for the one full share of the asset. Thus, the forward price has been immediately reimbursed. No additional dividend income is received until time T.

The cost of the initial purchase of $\left(1 - \frac{D}{S(0)}e^{-rt_0}\right)$ shares is

$$\left(1 - \frac{D}{S(0)}e^{-rt_0}\right) \cdot S(0) = S(0) - De^{-rt_0},$$

and the cost of the forward purchase price for the "missing" $\frac{D}{S(0)}e^{-rt_0}$ shares was exactly matched by the dividend payment. The end result—one share at time T with no "loose" dividend income along the way—is the same as that of a prepaid forward purchase. Hence, the prepaid forward price is the same as the price $S(0) - De^{-rt_0}$ of purchasing $\left(1 - \frac{D}{S(0)}e^{-rt_0}\right)$ shares. Statement (b) follows by making the appropriate time-value-of-money adjustment (multiply F^P by e^{rt} to get F), and the argument given here for the pricing formulas in the single dividend case generalizes easily to justify Statement (c).

□

Example 1.11. Consider a stock that pays dividends of \$3 at times $t = .25$ and $.75$, with no other dividend payments occurring during $t \in [0, 1]$. The current price of the stock is $S(0) = 35$. The continuously compounded risk-free rate is 5%.

(a) Find the forward and prepaid forward prices for delivery of the stock at time $t = 1$.

(b) If an investor enters a forward contract at time 0, what is the value of that forward at time $t = .6$ if $S(.6) = 35.10$?

Solution.

(a) The forward price is

$$F = S(0)e^{.05(1)} - \underbrace{3e^{.05(.75)}}_{t=.25 \text{ dividend}} - \underbrace{3e^{.05(.25)}}_{t=.75 \text{ dividend}} \approx 30.64;$$

that is, the time $t = 1$ value of $S(0)$ minus the time $t = 1$ value of the two dividends. The prepaid forward price is similar, but with cash amounts valued at time $t = 0$:

$$F^P = S(0) - \underbrace{3e^{-.05(.25)}}_{t=.25 \text{ dividend}} - \underbrace{3e^{-.05(.75)}}_{t=.75 \text{ dividend}} \approx 29.15$$

Observe also that the relationship $F^P = F \cdot e^{-rT}$ is satisfied by these prices.

(b) The price of a "new" forward contract at time $t = .6$ with the same $t = 1$ delivery date is

$$F = S(.6)e^{.05(.4)} - \underbrace{3e^{.05(.25)}}_{t=.75 \text{ dividend}} \approx 32.77.$$

The contract that was written at time $t = 0$ evidently offers a cheaper delivery price. The value of the original forward contract at time .6 is equal to the time $t = .6$ value of the $(32.77 - 30.64)$ difference in forward delivery prices:

$$(32.77 - 30.64)e^{-.05(.4)} \approx 2.088$$

□

1.2.4 Forward Contracts Involving Commodities; Additional Terminology

Forward contracts can also be written on commodities such as precious metals and livestock. Determining the forward price for future delivery of a commodity is similar to considerations involving underlying assets such as stocks and indices, but there is one key difference.

Holding a dividend-paying investment asset, such as a stock, during $[0,T]$ will provide income at the dividend rate δ, and the $e^{-\delta T}$ factor in the forward price $S(0)e^{(r-\delta)T}$ serves to remove the cost of this dividend income (which will not be received if a forward is used to purchase the stock) from the price of the asset.

By contrast, possessing a commodity—gold, grain, cattle, pork bellies—during $[0,T]$ pays no dividend to the owner but rather incurs costs to the owner. Gold must be stored and secured, grain must be kept dry, and those heads of cattle and pork bellies aren't going to feed themselves. Consider a situation in which the cost of housing and maintaining the commodity is proportional to the value of the amount owned of the commodity, so the cost of storing the commodity during a very short time interval $[t, t + dt]$ is $\lambda \cdot S(t) \cdot dt$ for some rate λ.

Whereas a forward contract on a stock involves reducing the stock's price by a factor of $e^{-\delta T}$ to "compensate" for missing out on dividends, a forward contract on a commodity should cost *more* than outright purchase of the asset. This is because the owner of a commodity forward has avoided the costs of storing the gold, feeding the cattle, etc. One can argue as we did in Theorem 1.8 that, in the absence of other factors (see next paragraph), the forward price (paid at time T for delivery at time T) on a commodity is

$$F = S(0)e^{rT}e^{\lambda T} = S(0)e^{(r+\lambda)T}.$$

An additional consideration that can affect the forward price is that there may be value associated with the convenience of holding an asset. A company that produces breakfast cereals, for example, may need to have a supply of grain on hand in order to ensure smooth production. Suppose that the value derived from the convenience of holding $S(t)$ worth of grain, say, during $[t,\ t + dt]$, is $c \cdot S(t) \cdot dt$ for some rate c. This rate c is called the **convenience yield**. In this setting, the forward price is

$$F = S(0)e^{rT}e^{\lambda T}e^{-cT} = S(0)e^{(r+\lambda-c)T},$$

the factor of e^{-cT} eliminating the value of the convenience of holding the asset from the price for forward delivery.

If you borrow money at rate r to fund the purchase of an asset, the interest paid at a rate r represents a cost. This cost may be offset by dividend income from an investment asset; for commodities, costs may be increased by the cost of storage but offset by convenience yield. The resulting net cost (expressed as a rate or percentage of the asset's value) is known as **cost of carry**. Thus, for a forward contract on a stock that pays dividends at continuous rate δ, the cost of carry is $r - \delta$. For a forward on a commodity with storage costs incurred at rate λ and convenience yield c, the cost of carry is $r + \lambda - c$.

Short selling an investment asset

Another bit of terminology relates to **short selling**. A short sale is an arrangement in which an asset is borrowed so that it can be sold to another party. Consider the short sale of a stock that pays dividends at rate δ. The party from whom a stock is borrowed will want to be compensated for the missing dividend income (while the stock is loaned out) at rate δ. This rate of compensation is known as the **lease rate**.

Commodity short sales

For a commodity short sale, the party from whom the commodity is borrowed will want to be compensated for any inconvenience associated with not having access to the asset, but this amount will be offset by savings (enjoyed by the lender of the commodity) from not having to store the commodity. As with lease rates on assets providing investment income, the **lease rate** for a commodity is defined to be overall the rate of compensation paid to the party who loaned the asset in a short sale, which would be $c - \lambda$. Note that in the case of both types of assets (investment assets and commodities), we have the following relationships:

$$\text{Cost of carry} = r - (\text{lease rate})$$
$$\text{Forward price} = S(0)e^{(\text{cost of carry})\times t} = S(0)e^{(r-(\text{lease rate}))\times t}$$

Forwards on currencies

As mentioned in the introduction to this chapter, one function of forward contracts can be to eliminate risk associated with future transactions that involve multiple currencies. Forward contracts can be written using a currency as the underlying asset. We will explore this in more detail in Section 1.7.

1.2.5 Payoff and profit for long and short positions in forward contracts

In a forward contract, one party (the owner of the forward contract) is obligated to pay the forward price F on the delivery date T; that party is contractually guaranteed to receive the underlying asset, whose value at that time will be $S(T)$ (a value which is not knowable in advance). The investor in this position—paying the forward price to receive the asset—is said to have taken a **long position** in the forward contract. The net exchange of value at time T, from perspective of the investor in the long position, is $S(T) - F$. This quantity is called the **payoff** associated with the long position in the forward contract.

The counterparty in this situation has exactly the opposite obligation and reward: This party must give up one share of the underlying at time T (worth $S(T)$) and will receive F at that time as compensation. The party in the position of delivering the asset is said to have the **short position** in the forward contract. From the viewpoint of the party who has the short position, the **payoff** is $F - S(T)$.

Example 1.12. The spot price for one share of a stock is $S(0) = 275.55$ and the stock does not pay dividends; the continuously compounded risk-free rate is 3%. Compute the payoff for both long and short positions in a one-month forward contract on this stock (a) if $S(1/12) = \$260$; (b) if $S(1/12) = \$300$.

Solution.

$$F = S(0)e^{(r-\delta)\times 1/12} = 275.55e^{(.03-0)\times 1/12} = 276.24.$$

Thus, the time$-1/12$ payoff for a long position in the forward contract is $S(1/12) - 276.24$, and the payoff for the short position in the forward is $276.24 - S(1/12)$. So

(a) if $S(1/12) = \$260$,
then the long forward position has payoff $260 - 276.24 = -\$16.24$,
and the short forward position has payoff $276.24 - 260 = \$16.24$;

(b) if $S(1/12) = \$300$,
then the long forward position has payoff $300 - 276.24 = \$23.76$,
and the short forward position has payoff $276.24 - 300 = -\$23.76$

□

Arbitrage and synthetic forwards

If the forward price is different than $S(0)e^{(r-\delta)T}$, then there may be an arbitrage opportunity.

Example 1.13.

A stock currently trades at \$275.55. This stock does not pay any dividends. An arbitrageur who can borrow or lend at the risk-free rate $r = .03$ notices an opportunity to take either a long or short position in a 3-month forward contract with forward price of \$280.00. Describe actions that can be taken by the arbitrageur to effect arbitrage profit if there are no other costs involved.

Solution.

The forward price \$280.00 is higher than the "theoretical" forward price of $275.55\ e^{(r-\delta)(.25)} = 275.55\ e^{.03\times.25} \approx \$277.62$. Therefore, the arbitrageur should take a short position in the overpriced forward contract and borrow \$275.55 (which will be repaid with interest at time $T = .25$) to purchase one share. There is no net cost to the arbitrageur at time 0 to setting up the overall position.

At time $T = .25$, the payoff for the arbitrageur is as follows:

$$\begin{pmatrix} \text{value of} \\ \text{purchased stock} \end{pmatrix} - (\text{repayment of loan}) + \begin{pmatrix} \text{payoff of short} \\ \text{forward position} \end{pmatrix}$$
$$=$$
$$(S(T)) - (275.55e^{.03\times.25}) + (280 - S(T))$$
$$=$$
$$280 - 275.55e^{.03\times.25} \approx \$2.376$$

No matter what happens to the price of the stock over the next three months, the arbitrageur will receive net income of \$2.376, having invested nothing at the outset.

$\square$

Part of the arbitrage strategy in Example 1.13 was to employ a **synthetic forward** to offset the short position in the "overpriced" forward contract. A synthetic forward is an investment position that recreates the payoff structure of a forward contract using some alternative combination of other assets/derivatives. In this case, the payoff structure of a forward contract was replicated via a risk-free bond (view selling/taking a short position in a risk-free bond as the borrowing mechanism) and using the borrowed funds to purchase one share of the underlying asset. The results of these two steps involved in this synthetic forward are (1) The investor has zero net cash inflow/outflow at time 0, because the borrowed funds are immediately invested in the stock purchase, and (2) at $T = .25$, the investor pays out $S(0)e^{(r-\delta)\times .25}$ and walks away with one share. The overall cash flows at times 0 and .25 are no different from those with a forward contract that has the "correct" theoretical $S(0)e^{(r-\delta)\times .25}$ delivery price.

For an *underpriced* forward contract (that is, if the contractual forward price is below $S(0)e^{(r-\delta)T}$), the arbitrageur would take a long position in the forward contract and create a short synthetic forward by selling one share of the asset and lending out the income (the lending mechanism is a long position/purchase of a risk-free bond).

Transaction costs affecting profit calculation

Of course, the real world is not quite so ideal. Most investors are subject to transaction costs or fees such as commissions, and the rate at which an investor could actually borrow money might be somewhat higher than the rate at which money could be loaned out (the latter being possible if the investor has a risk-free bond available to sell).

Moreover, the price at which an investor can purchase one share of stock (the ask price) from an exchange is, at any given moment, a little higher than the price at which one share could at that time be sold back to the exchange (the bid price). This fact of life is referred to as the **bid-ask spread**.

Example 1.14. The current bid and ask prices for one share of a stock are \$275.50 and \$275.60, respectively. An investor can also take either a long or short position in a 3-month forward contract with forward price \$280.00 on one share of the stock. There is a \$1 transaction fee for making a stock trade as well as a \$1 fee for taking a position in the forward contract—these commission fees are paid at time 0. The continuously compounded interest rates at which the investor can borrow or lend are 4% and 2%, respectively. Consider the strategy "short position in the forward contract + long synthetic forward" employed in the solution to Example 1.13. Does this strategy result in arbitrage profit?

Solution.

Under the strategy described above, the investor borrows funds to purchase one share at time 0. The purchase price for this share is the ask price \$275.60. The investor also enters a short position in the forward contract. The stock purchase and the initiation of the forward contract each have associated costs to the investor of \$1, for a total of \$2 in transaction costs. Thus, the investor borrows $(275.60 + 2) = \$277.60$ at time 0 at 4% interest. Overall, the investor experiences no net income or outflow of cash at time 0.

At time $T = .25$, the payoff for this strategy is as follows:

$$\begin{pmatrix} \text{value of} \\ \text{purchased stock} \end{pmatrix} - (\text{repayment of loan}) + \begin{pmatrix} \text{payoff of short} \\ \text{forward position} \end{pmatrix}$$
$$=$$
$$(S(T)) - (277.60e^{.04\times .25}) + (280 - S(T))$$
$$=$$
$$280 - 277.60e^{.04\times .25} \approx -\$.39$$

The strategy results in a loss. The seemingly mispriced forward contract was not sufficiently mispriced to overcome the bid-ask spread, the interest rate spread, and the transaction costs.

□

1.3 Futures Contracts

A forward contract between two parties has an inherent risk: Suppose the value of the underlying asset becomes especially low at delivery time—much lower than initially anticipated and much lower than the contracted forward price. There is some risk that the party responsible for paying the forward price could refuse to pay, in violation of the terms of the contract.

Futures contracts are similar to forward contracts but with features that minimize the risk that an investor might fail to pay the delivery price. A futures contract specifies an asset, delivery date, and a **futures price**, which is the agreed-upon price for the asset to be paid on the delivery date. Like a forward contract, the futures contract requires the holder to pay (via mechanics described below) a contractually specified delivery price on the delivery date to receive the underlying asset on that date. The underlying assets for a futures contract can be commodities (such as gold, natural gas, or livestock) or investment assets like stock market indices.

1.3.1 Futures contract terminology: Margin accounts and the marking-to-market process

Unlike forward contracts, futures contracts are traded on exchanges using standardized contracts. To reduce the risk of nonpayment of the delivery price, an investor in a futures contract must deposit funds into a **margin account**. Each day, at the end of trading, the exchange deposits (or withdraws) funds into the account if the underlying asset has increased (respectively, decreased) in value. The amount of the deposit or withdrawal is equal to the overnight change in the price of futures contracts with the same delivery date. This daily adjustment of the margin account is called **marking to market**.

The futures contract specifies a minimum margin account balance known as a **clearing margin** or **maintenance margin**. Whenever the underlying asset has performed poorly enough to deplete the margin account below the clearing margin, the exchange can issue a **margin call** to the investor, requiring additional funds to be deposited into the margin account.

At any time prior to the delivery date, the holder of a futures contract may close out the position, having already realized any gains or losses on a day-by-day basis via adjustments to the margin account. Futures contracts on stock indices are settled in cash, and it is usual for futures contracts on commodities to be closed out prior to the delivery date.

1.3.2 Evaluating an investor's margin balance as the underlying asset's value changes

Example 1.15. Consider an investor who enters a 6-month futures contract on 10 shares of a stock index. The index is currently trading at $S(0) = 200$, where $t = 0$ represents January 1st. The index pays dividends continuously at a rate of $\delta = 5\%$ per year and the risk-free interest rate is $r = 6\%$. The futures price for delivery of 10 shares of the index on July 1st (i.e. the delivery price that would be paid at $t = .5$) is

$$\begin{aligned} F &= 10 \cdot S(0) e^{(r-\delta)(.5)} \\ &= 10 \cdot 200 e^{(.06-.05)(.5)} \approx 2010.03 \end{aligned}$$

This amount \$2010.03 is the published futures price. Note that the futures price is quoted as a price that will be paid in the future—we would not need to "adjust" a quoted futures price \$2010.03 to remove dividends or change its valuation date, as these calculations have already occurred "behind the scenes" and are incorporated into the \$2010.03 futures price.

The contract requires the investor to deposit \$2010.03 into the margin account at time $t = 0$ and specifies that a maintenance margin of \$1500 must be kept in the account.

Table 1.1 shows the futures price (quoted on several dates at close of trading day) for delivery on July 1st of 10 shares of the index and lists what adjustments are made by the exchange to the margin account in the process of marking to market. Many margin accounts credit interest to the investor; we assume for this example that funds in the margin account earn the risk-free rate of 6%.

Table 1.1: Margin Account for Example 1.15

Date	**Futures price at close of trading for delivery on July 1**	**Resulting adjustment to margin account**	**New balance of margin account**
Jan. 1 (the moment at which contract was written)	\$2010.03 (moment that contract was written)		Initial account value \$2010.03

Table 1.1: Margin Account for Example 1.15 (continued)

Jan. 1 (at close of trading)	$2011.00 (increase of $.97)	add .97	$2011.00
Jan. 2 (close)	$2009.30 (decrease of $1.70)	add interest $\left(2011 \times e^{.06/365}\right)$ then subtract 1.70	$2009.63
Jan. 3 (close)	$2003.85 (decrease of $5.45)	add interest $\left(2009.63 \times e^{.06/365}\right)$ then subtract 5.45	$2004.51
Jan. 4 (close)	$2006.11 (increase of $2.26)	add interest $\left(2004.51 \times e^{.06/365}\right)$ then add 2.26	$2007.10

The index did not perform well between January 4 and April 2, and the adjustments to the margin account brought the balance below the $1500 maintenance margin. This caused the broker to issue a margin call to the investor, requiring a deposit of $510 into the margin account in order to restore the balance to a level that will cover the delivery price:

Table 1.2: Margin Account for Example 1.15, continued

Date	**Futures price at close of trading for delivery on July 1**	**Resulting adjustment to margin account**	**New balance of margin account**
April 2 (close)	$1488.33	...	$1514.07
April 3 (close)	$1471.32 (decrease of $17.01)	add interest $\left(1514.07 \times e^{.06/365}\right)$ then subtract 17.01 to get $1497.31; investor must deposit $510 in response to margin call	$1497.31 +$510 =$2007.31
...	...	...	...

Throughout the life of the futures contract, the investor has the option to close out his or her position at any date, walking away with the account balance and no further obligation to purchase the asset. Any losses or gains to date will have been realized by the daily adjustment to the margin account.

In situations where the index has performed well, the margin account balance may have grown well beyond the initial \$2010.03 balance. The investor may opt to withdraw any extra balance above \$2010.03.

■

1.3.3 Comparing forwards and futures

Any two parties can enter a forward contract; sometimes this is done "over the counter" through online marketplaces. There is much flexibility in how a forward contract might be drawn up, and these can be constructed to suit needs that are particular to an individual investor. More variety in choice of underlying assets is possible with a forward contract. This variety can in some instances make it difficult to close out a position in a forward contract: there may not be much marketplace demand to take over contracts written on unusual assets. Futures contracts, on the other hand, are highly standardized and only sold on exchanges; it is easy to close out a position in a futures contract as described above in Example 1.15.

As mentioned above, forward contracts can involve much more credit risk than futures contracts. If the underlying asset suddenly loses much of its value as the delivery date approaches, that loss will be absorbed all at once (at delivery) by the holder of the contract. The forward contract itself offers little protection to the writer of the contract (the seller of the asset under the contracted arrangement) if the opposite party is tempted to abandon his or her contractual obligations to pay the agreed-upon delivery price.

In a futures contract, this credit risk is largely mitigated through the daily process of marking to market, which eliminates to some extent the temptation for an investor to abandon his or her contractual obligations. Whereas a large loss in a forward contract would be realized all at once (at delivery), any particular day's losses in a futures contract would be settled up on the same day via adjustment to the margin account.

Table 1.3: Comparison of Forward and Futures Contracts

	Forward contract	Futures contract
Credit risk	Investor may have incentive to walk away from paying contracted delivery price.	Very little credit risk: any particular day's losses are realized same-day.

Table 1.3: Comparison of Forward and Futures Contracts (continued)

Timing of profits/losses	Entire profit/loss realized at settlement.	Profits/losses realized same-day through daily settlement (marking to market) of margin account.
Margin account / daily marking to market	No	Yes
Nature of contract	Private contract between two parties.	Standardized terms; traded on exchanges.
Closing out contract	Asset delivery or cash settlement of $S(T) - F$ are both common; can be difficult to close out position.	Closing out contract prior to maturity is more common than asset delivery.

1.4 Puts and Calls

In this section, we will introduce puts and calls, which are financial derivative instruments that provide an investor with a right to sell (put) or a right to purchase (call) a share of an asset. Because the owner of a call or put may choose whether or not to use the contractual right to buy or sell, these contracts are said to be **option contracts**.

Investors, market makers, and risk managers use options as building blocks to construct portfolios that accomplish many risk management and investment goals. Options allow an investor who has a long position in a particular asset to lock in the gains on that position. Options can likewise limit losses for an investor who takes a short position in an asset. Puts and calls may also be used to get exposure to the price movements of a security without actually having to hold the security.

Options are fundamentally different from forwards and futures in the following way. The owner of an option is not obligated to exercise the rights provided under the contract. Thus, it is not guaranteed that the option will be used, whereas the contractual terms of a forward contract *require* the exchange of the forward price in return for the asset. (This exchange is automatically enforced as well for a futures contract through the daily adjustments to the margin account.)

1.4.1 Definitions and terminology

A **call** (on a share of a stock or other underlying asset) is a contract that gives its owner the right to purchase the asset at some point (or at various points) in the future at a contractually specified price. The contracted price at which the asset may be purchased is called the **exercise price** or **strike price**. To clarify: The owner of the call is not required to exercise the right to purchase the underlying asset; the call is a right to purchase, not an obligation. The "opposite party" is sometimes referred to as the **writer** of the call option—if the owner of the call exercises the option by paying the strike price, the writer is contractually obligated to supply a share of the underlying asset.

The party wishing to obtain a call contract must pay the writer of the option a fee called a **premium**—this is the price of the contract itself. The party who purchases the call is said to take a **long position** in the call; the counterparty who writes (sells) the option is said to have a **short position**.

Additional adjectives describe the timing at which a call option may be exercised:

- A **European** call gives its owner the *right to purchase* the asset at the strike price only at the expiration date of the call contract.
- An **American** call allows its owner to purchase the asset at the strike price at any time prior to the expiration of the contract.

- A **Bermudan** call option includes a list of contractually specified dates at which the purchase of the underlying asset at the contracted strike price may occur.

The adjectives European, American, and Bermudan are sometimes said to specify the **style** of an option. To clarify: An American or Bermudan option may be exercised at most one time.

Consider a call on a stock. The call would only be exercised in situations where the current stock price is higher than the strike price. When a call is exercised, the difference between the value of the stock and the strike price is called the **payoff** for the owner of the call option. If the call expires without being exercised, we say that the payoff is zero. The **profit** resulting from owning a call option is equal to the payoff minus the value (at exercise) of the option premium that was paid at time 0. Let us illustrate this terminology:

Example 1.16. At time 0, you pay a premium of \$5 to purchase a six-month European call on a stock. The call has strike price \$60. The risk-free rate is $r = .03$. Compute the payoff and profit (a) if $S(.5) = \$66.50$; (b) if $S(.5) = \$59.11$.

a) If, in six months, the stock's price is \$66.50, then the payoff and profit from the call are as follows:

$$\text{Payoff} = 66.50 - 60 = \$6.50$$
$$\text{Profit} = 6.50 - 5e^{.03(.5)} \approx \$1.424$$

b) If, at time $t = .5$, the stock's price is \$59.11, the call expires worthless. There is no reason to use the right-to-buy at the \$60 strike price when the stock can be obtained for less money without using the option. The call expires worthless, and the profit associated with the purchase of the call is negative:

$$\text{Payoff} = \$0$$
$$\text{Profit} = 0 - 5e^{.03(.5)} \approx -\$5.076$$

■

Various types of "moneyness" for call options. At any time for which the underlying asset has a value that is greater than a call option's strike price, one says that the option is (at that time) **in the money**. For an in-the-money option for which the difference between the asset price and strike price is quite large, one can emphasize this by saying that the option is **deep** in the money.

If an option is "in the money" on a date for which exercise is permitted, then it is possible for the owner to obtain a positive payoff. Any time for which the underlying asset has a value that is *less* than a call option's strike price, the option is said to be **out of the money**. Thus, in part (a) of Example 1.16, the option expired in the money, and in part (b), the option was out of the money at expiration.

Finally, if one purchases a call option with strike price chosen to be equal to the current spot price $S(0)$ for the underlying asset, the option is said be an **at-the-money** option.

A **put** (on a share of a stock or other underlying asset) is a contract that gives its owner the *right to sell* the asset at some point (or at various contractually specified points) in the future at a contractually specified price. If a put option is exercised, it cannot be exercised a second time.

Much of the terminology that we have used to describe various aspects of call options is also used to describe put options. The contracted price at which the asset may be sold in a put contract is referred to as the **exercise price** or **strike price**. The cost to obtain a put option is called the **premium** and is paid to the **writer** of the contract. The owner of a put option may choose to exercise the option by requiring the writer to purchase the underlying asset from the put owner at the strike price. The owner of a put is *not* required to exercise the option—a put option provides an opportunity to sell the underlying asset; it does not require the put owner to sell the asset.

As with call options, the party who has purchased a put has taken a **long position** in the put. The counterparty who has written the put is said to have a **short position** in the put.

Put contracts may be written in the same three styles as call contracts: European puts may be exercised by their owners only at the expiration date of the contract, American puts may be exercised at any time until expiration, and Bermudan puts give specific points in time at which exercise is permitted.

The notions of payoff and profit associated with owning a put option are defined in the same manner as with calls. A put contract would only be exercised if the strike price (received by the owner of the put) exceeds the value of the asset that would be sold at that price. In this case, the **payoff** (for the owner of the put) is the difference between the strike and the asset value. If exercise does not occur prior to expiration, we say that the payoff is zero. The put owner's **profit** is defined as payoff less the at-exercise-value (or at-expiration-value) of the put premium.

Example 1.17. You purchase a six-month European put on a stock with strike price \$60. You pay a premium of \$5 at time $t = 0$ to obtain the put. The risk-free rate is $r = .03$. Compute the payoff and profit (a) if $S(.5) = \$58$; (b) if $S(.5) = \$64$.

a) If the stock price at time $t = .5$ is \$58, then the payoff and profit are the following:

$$\text{Payoff} = 60 - 58 = \$2$$
$$\text{Profit} = 2 - 5e^{.03(.5)} \approx -\$3.076$$

Note that a negative profit can still result even when the payoff is positive. In this illustration, the payoff was not sufficient to recover the cost of purchasing the put. Clearly, the lower the stock price at time $t = .5$, the bigger the payoff resulting from selling a share at \$60.

b) If the stock price at time $t = .5$ is \$64, then the put option expires worthless:

$$\text{Payoff} = \$0$$
$$\text{Profit} = 0 - 5e^{.03(.5)} \approx -\$5.076$$

Note that option premiums are quoted at time 0 when the option is purchased; payoff and profit are valued at the time of exercise or at expiration (if exercise does not occur). ■

Moneyness for puts. For a put options to have a positive payoff, it is necessary for the underlying asset to have a value that is less than the strike price. Hence, one says (regardless of whether exercise is permitted at the time) that a put option is **in the money** if the underlying asset is worth less than the put's strike price or **out of the money** if the asset is worth more than the strike price. As with calls, if one buys a put option with strike price set equal to $S(0)$, he is said to have purchased an **at-the-money** put.

1.4.2 Payoff and profit functions: European call options

Long position in a European call. Consider the owner of a European call option on a stock with strike price K and expiration date T—the investor has a long position in this call, providing the right to purchase the stock at K. Because the call is a European option, this right may be exercised only on the expiration date.

If the stock price $S(T)$ at expiration is greater than the strike, the owner of the call will exercise the option, receiving stock worth $S(T)$ and spending K. If the contract is settled in cash, the amount that the call owner would receive is $S(T) - K$, the cash equivalent of receiving a share by spending K. On the other hand, if $S(T) < K$, no one would spend the strike to receive a share that could be purchased more cheaply without using the call; the payoff in this situation is 0.

We see that a call never has a negative payoff—the smallest possible payoff is 0. We can express the call's payoff in two equivalent ways:

$$\text{Long call payoff} = \max\{S(T) - K, 0\} \tag{1.25}$$

$$\text{Long call payoff} = \begin{cases} S(T) - K & \text{if } S(T) \geq K \\ 0 & \text{if } S(T) < K \end{cases} \tag{1.26}$$

Formula (1.25) is especially useful for implementing pricing models in spreadsheet software. Formula (1.26) lends itself to understanding the payoff for a portfolio that includes K-strike calls. Let us take the view that the payoff for a K-strike European call is a function whose "input variable" is underlying asset's value $S(T)$ at the expiration date. We would like to graph the possible payoffs resulting from the various possible stock prices $S(T)$. Formula (1.26) allows us to make some helpful observations concerning the graph of this payoff function:

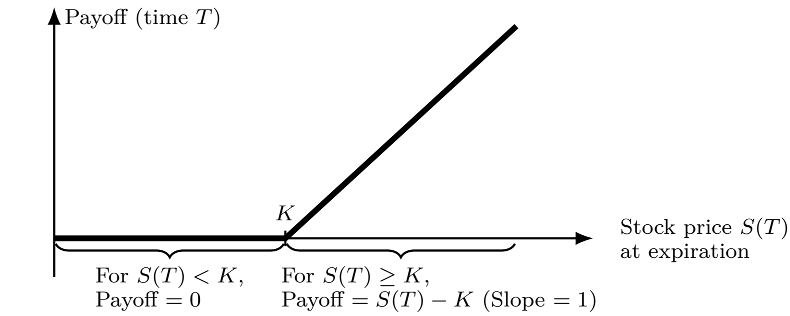
- The graph of the payoff as a function of $S(T)$ looks like the graph of the function

$$f(s) = \begin{cases} s - K & \text{if } s \geq K \\ 0 & \text{if } s < K \end{cases}. \tag{1.27}$$

- Thus, the payoff for a European call is a continuous function of $S(T)$, and
- the graph of this function has slope 0 for stock prices below K and slope 1 for stock prices above K (the coefficient of s in (1.27) is 1 for $s > K$).

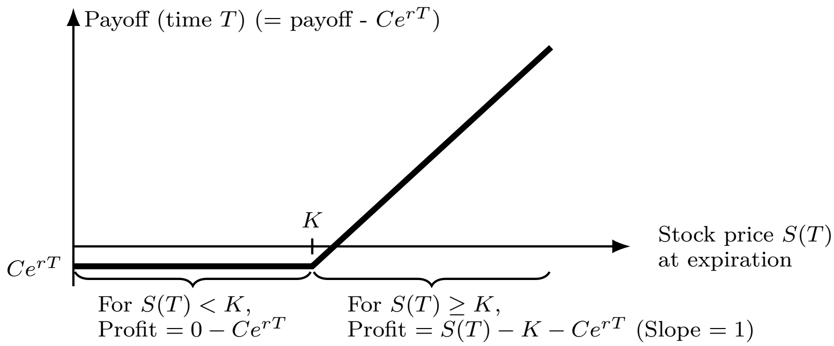
Hence, we have the following graph (Figure 1.1) of payoff as a function of time-T stock price for the owner of a European call option.

Figure 1.1: **Payoff at expiration for long position in a European call**



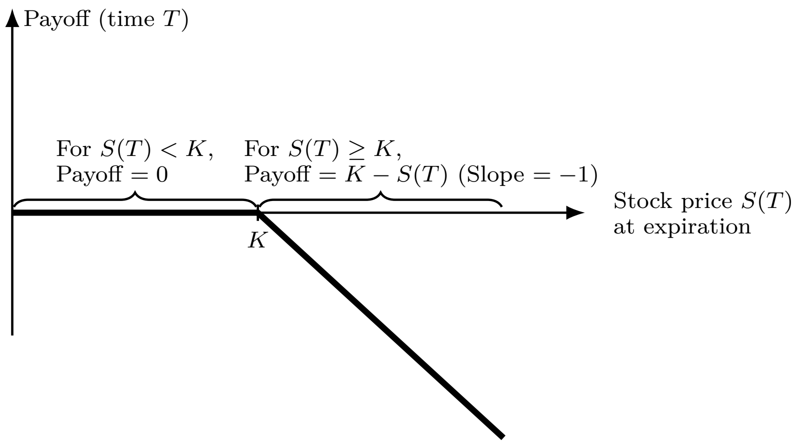
Let C denote the option premium for the call option. Because profit values are obtained by subtracting the value-at-expiration Ce^{rT} of the premium from the payoff, we can easily modify our payoff graph to construct the graph of the profit function, plotting possible profit amounts against stock price at expiration: simply shift the graph of the payoff function down by Ce^{rT} units to obtain the payoff graph (Figure 1.2):

Figure 1.2: **Profit at expiration for long position in a European call**



Short position in a European call. Consider the perspective of the counterparty who has written (i.e. sold) a call. Recall that the party who has sold the option is said to have a short position. The conditions under which the call is exercised by its owner have not changed, but from the call writer's perspective, the cash flows go in the opposite direction when compared to the call owner's cash flows. So the payoff *for the short position in the call* is $K - S(T)$ if $S(T) \geq K$ (the owner of the option will enforce his or her right to buy the underlying asset, so the writer of the call receives the strike price from the holder of the call option and in return gives up one share of the underlying asset); the payoff is 0 if $S(T) < K$. See Figure 1.3.

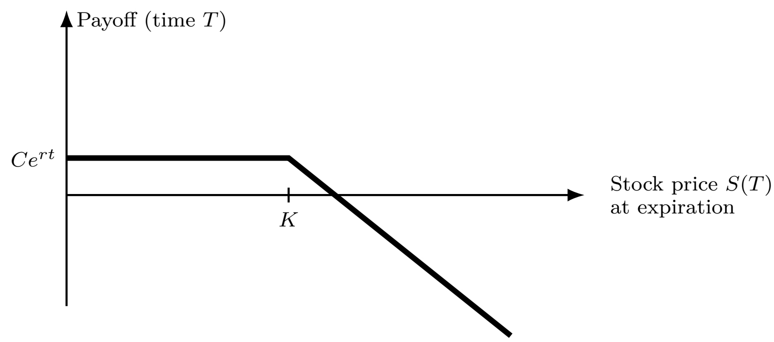
Figure 1.3: **Payoff at expiration for short position in a European call**



$$\text{Short call payoff} = \begin{cases} K - S(T) & \text{if } S(T) \geq K \\ 0 & \text{if } S(T) < K \end{cases} \tag{1.28}$$

Because the call's seller has been paid the option premium C, the payoff graph (Figure 1.4) for the short position can be obtained by shifting the payoff graph up by Ce^{rT} units.

Figure 1.4: **Profit at expiration for short position in a European call**



1.4.3 Payoff and profit: European put options

Let's repeat our discussion above, this time for a K-strike European put on some stock with expiration at time T. Puts are only exercised if the stock price is lower than the strike price at which the stock would be sold through the put option. If exercised, the owner of the put will give up a share of the stock (whose value at exercise is $S(T)$) and receive a payment of K; if settled in cash (as opposed to actually transferring shares of the stock), the put owner would receive the difference $K - S(T)$.

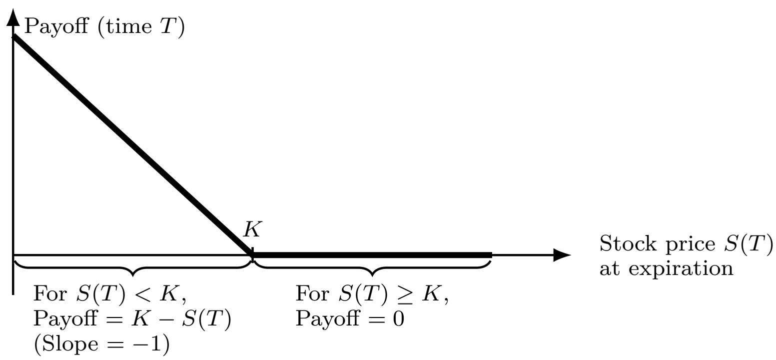
As with calls, the payoff formula for a European put can be expressed using "max" notation for spreadsheet entry as well as using piecewise-defined function notation:

$$\text{Long put payoff} = \max\{K - S(T), 0\} \tag{1.29}$$

$$\text{Long put payoff} = \begin{cases} K - S(T) & \text{if } S(T) \le K \\ 0 & \text{if } S(T) > K \end{cases} \tag{1.30}$$

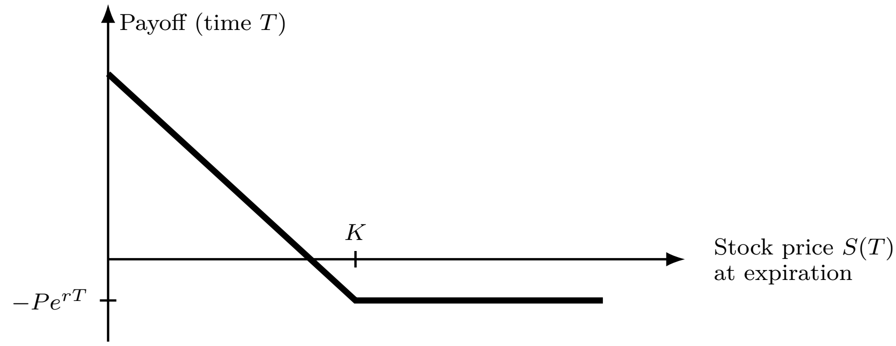
We construct the payoff graph by plotting possible payoffs on the vertical axis and possible stock prices $S(T)$ on the horizontal axis, again viewing payoff as a function of $S(T)$. Observe from Equation (1.30) that the put payoff is a continuous function of S whose slope (coefficient of $S(T)$) is -1 for stock prices $S(T)$ that are smaller than K, and that put payoff is zero for larger stock prices. These observations allow you to quickly visualize the graph of the payoff function (see Figure 1.5) for a European put:

Figure 1.5: **Payoff at expiration for long position in a European put**



If the put option was purchased for an option premium P, then profit function's graph (Figure 1.6) looks like the payoff graph shifted down by the time-T value Pe^{rT} of the option premium:

Figure 1.6: **Profit at expiration for long position in a European put**



As with call options, the cash flows for a short position in a put option (i.e. the cash flows from the perspective of the seller of the option) are the opposite of the cash flows in the long position (i.e. the cash flows from the perspective of the owner of the option). The owner of the put—the owner of the right to sell at a fixed price K—will utilize that right in precisely those situations for which $S(T) < K$. In those situations, the holder of the option will sell one share to the writer of the put. The writer of the put—the party with the short position—is therefore required to give up K, receiving $S(T)$ in return. The put is not exercised by its owner in the case $S(T) > K$. The put writer received the option premium P at time 0 (when the put option was sold), and so the profit (profit is always valued at expiration, i.e. at time T) for the put writer's short position is Pe^{rT} greater than the payoff.

Summary. Table 1.4 summarizes the payoff formulas for long and short positions in European call and put options. The piecewise-defined versions (i.e. "if $S(T) < K$ …"; "if $S(T) > K$…") of the payoff formulas will be particularly useful in helping us to construct payoff graphs for various combinations of long and short puts and calls.

It is helpful to keep in mind that whether an investor's position in a call (a right to purchase) is long or short, that call would only be exercised if the resulting purchase would be a "good deal", that is, if the stock price is higher than the strike K. Likewise, whether an investor's position in a put option is long or short, the party holding the put option (a right to sell) would only exercise it if it is a "good deal" for them, that is, if the market price of the asset is lower than the price that could be obtained by selling the asset at the strike K.

Table 1.4: Payoffs for long and short positions in European options (strike K, expiration T)

Investor's position	Payoff if $S(T) < K$	Payoff if $S(T) > K$	Spreadsheet formula
Long call	0	$S(T) - K$	$=\max(S(T) - K, 0)$
Short call	0	$K - S(T)$	$= -\max(S(T) - K, 0)$
Long put	$K - S(T)$	0	$=\max(K - S(T), 0)$
Short put	$S(T) - K$	0	$= -\max(K - S(T), 0)$

Note that all of the "spreadsheet-style" formulas involve the MAX function: the party who owns the option chooses whether to exercise such that their payoff is maximized. The negative sign in the short position formulas simply reverses which party gets a share of the underlying asset and which party pays the strike price.

1.4.4 Payoff for portfolios involving European options

In the next section of this chapter, we will explore various strategies that involve assembling a portfolio that includes put and call options. Let us consider a couple of examples to get used to how one analyzes payoff and profit for combinations of options.

Example 1.18. You enter the following positions in European options with expiration date one year (so $T = 1$):

- Purchase a 40-strike call (i.e. you are long a 40-strike call), and
- Sell a 50-strike call (i.e. you have written a 50-strike call; you are short a 50-strike call).

(a) Compute a piecewise-defined payoff formula for this option package. Then construct the graph of payoff as a function of $S(T)$.

(b) You are given that $r = .04$. Determine the profit function if the 40-strike call has a premium of 14 and the 50-strike call has premium 9.50. What modification to the payoff graph will produce the profit function's graph?

Solution.

(a) We will organize our computation of the portfolio's payoff function by using a chart (Table 1.5) that keeps track of each option's payoff in the cases $S(T) < 40$, $S(T) \in [40, 50]$, and $S(T) > 50$.

Table 1.5: Payoffs for bull spread of Example 1.18

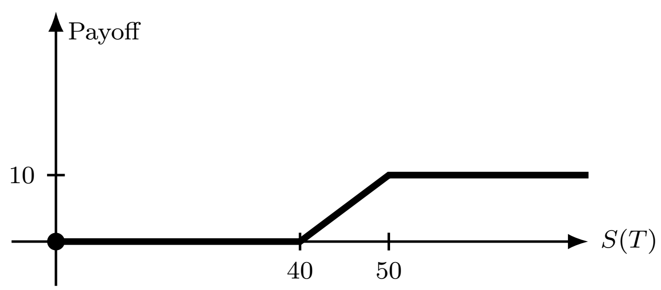
	$S(T) < 40$	$S(T) \in [40, 50]$	$S(T) > 50$
Long 40-call payoff	0	$S(T) - 40$	$S(T) - 40$
Short 50-call payoff	0	0	$50 - S(T)$
Payoff for portfolio	0	$S(T) - 40$	10

Let's walk through the entries in this table. If $S(T) > 40$, the 40-strike call allows the investor to spend 40 to receive one share, the value of which at exercise would be $S(T)$. Thus, the 40-strike call has payoff $S(T) - 40$ if $S(T) > 40$. The 40-strike call would not be exercised if $S(T) < 40$.

Now let's deal with the short 50-strike call payoff. The "opposite party" would exercise his or her right to buy the stock at the strike price 50 precisely in those situations where $S(T) > 50$. You (the seller of the call) will receive 50 but will have to supply one share of the stock at a cost of $S(T)$. Hence, the payoff for your short position in the 50-strike call is negative (equal to $50 - S(T)$) if $S(T) > 50$.

The entries in the "Payoff for Portfolio" row of the table are obtained by adding vertically within each column. It is now easy to construct the graph for the portfolio payoff as a function of the various possible values for $S(T)$. As we've noticed before, the payoff for a call option is a continuous piecewise-linear function of $S = S(T)$, with changes in slope occurring at the strike price. The same could be said for any sum of such functions. Hence, our payoff graph is continuous and looks like the zero function until we reach $S(T) = 40$, the graph is linear with slope $+1$ for $S(T) \in [40, 50]$, and the payoff is $+10$ if $S(T) > 50$.

Figure 1.7: Payoff for bull spread of Example 1.18



(b) At time 0, the cost of purchasing the 40-strike call is offset by the sale of the 50-strike call. The premium (i.e. cost) for this package of options is

$$14 - 9.50 = \$4.50$$

(One could choose instead to keep track of time-0 income, which would be $-14 + 9.50 = -\$4.50$, a net time-0 cash outflow.)

Profit is defined to be payoff minus value-at-exercise (or at expiration if option is not exercised) of time-0 costs; so to get the profit function (Table 1.6 below), we subtract $(14 - 9.50)e^{.04(1)} \approx 4.684$ from each of the three possible payoff formulas from Table 1.5:

Table 1.6: Profit for bull spread of Example 1.18

	$S(T) < 40$	$S(T) \in [40, 50]$	$S(T) > 50$
Profit for portfolio	$0 - 4.684$ $= -4.684$	$S(T) - 40 - 4.684$ $= S(T) - 44.684$	$10 - 4.684$ $= 5.316$

Hence, the graph of profit (not pictured) is obtained by shifting the payoff graph down by $4.5e^{.04(1)} \approx 4.684$.

□

Remark. The graph of the payoff function in Example 1.18 has the following features:

- It is continuous and piecewise-linear with three "sections",
- the center section has positive slope, and
- the other sections have zero slope.

A portfolio whose payoff has these features is called a **bull spread**. The adjective "bull" comes from the idioms "bear market"/"bull market", with a bull market being a favorable investment environment. Note the higher payoffs at the upper end of the interval $S(T) \in [40, 50]$. An investor might enter a bull spread to act on a belief that the price of a stock will experience a slight increase during the year.

Note that the sections of the payoff function and diagram for which $S(T) < 50$ are identical to the corresponding payoffs for a 40-strike call option. This particular bull spread allows the investor to obtain some (up to \$10) of the payoff offered by a 40-strike call, but at a lower initial option cost (\$4.50 for the spread vs. \$14 for the call). Thus, the investor who believes that the stock price will *not* experience a *dramatic* increase beyond the \$40-50 range by time $T = 1$ can use a bull spread to reduce the cost of capturing those payoffs from the 40-strike call that he or she believes to be the most likely outcomes.

Example 1.19. You enter the following positions in European options with expiration date one year:

- Purchase a 50-strike call (you are long a 50-strike call), and
- Purchase a 50-strike put on the same underlying asset.

Compute the payoff function for this portfolio, and graph the payoff function by analyzing slopes.

Solution.

The 50-strike call—a right to purchase the underlying—will be exercised at $T = 1$ if and only if that purchase would be a "good deal", that is, if and only if $S(T) > 50$. In that case, you will gain a share worth $S(T)$ and spend \$50. Payoff for the 50-call is therefore $S(T) - 50$ if $S(T) > 50$; payoff is 0 otherwise. This is recorded in the "long 50-call" row of Table 1.7 below.

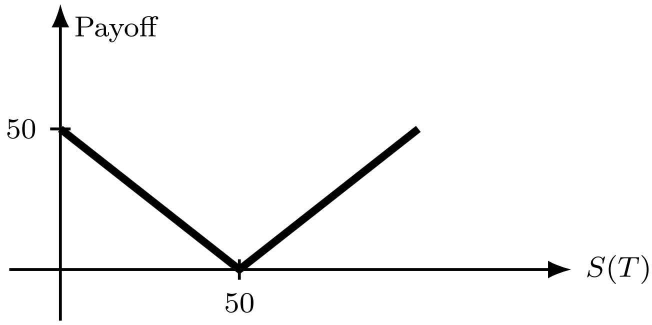
The 50-strike put—a right to sell the underlying at \$50—will be exercised if and only if it is advantageous to do so, that is, if and only if $S(T) < 50$. In that case you will give up a share worth $S(T)$ in exchange for receiving \$50, resulting in a net payoff of $50 - S(T)$. Payoff for the put is 0 if $S(T) > 50$. This gives the "long 50-put" row of Table 1.7.

Adding the various payoffs in each column of Table 1.7, we obtain the payoff function for this portfolio:

Table 1.7: Payoff for straddle of Example 1.19

	$S(T) < 50$	$S(T) > 50$
Long 50-call payoff	0	$S(T) - 50$
Long 50-put payoff	$50 - S(T)$	0
Payoff for portfolio	$50 - S(T)$	$S(T) - 50$

Figure 1.8: Payoff for straddle of Example 1.19



□

Remark. The graph for our payoff function in Example 1.19 has the following features:

- It is continuous and piecewise-linear,
- One section has slope -1, and
- The other section has slope $+1$.

A portfolio with these features is said to be a **straddle**. A straddle is constructed by taking identical positions (i.e. both long or both short) in a call and in a put on the same asset with the same strike price. An investor who believes the price of a stock to be very volatile—believing that the price at time 1 will be very different from \$50, say, but not knowing which side of \$50 is the more likely outcome—could arrange a straddle such as the one in Example 1.19 to speculate on that belief. The bigger the difference between $S(T)$ and \$50, the bigger the payoff for the investor.

1.4.5 *Payoff formulas for positions in stocks/indices and risk-free bonds*

In the coming sections, we will also need to deal with portfolios that involve borrowing and lending at the risk-free rate and portfolios that involve long or short positions in the underlying asset.

Payoff/profit for a bond. Suppose you spend B at time 0 to purchase a zero-coupon risk-free bond that you will redeem in the future at time T. (You are lending at the risk-free rate.) At time T, your payoff will be Be^{rT} The profit for the bond is obtained by subtracting the time-T value of the initial investment of B in the bond:

$$\text{Profit from bond purchase} = \text{Payoff} - Be^{rT} = Be^{rT} - Be^{rT} = 0.$$

No profit resulted, because the bond was a risk-free investment; the marketplace does not support arbitrage opportunities. Thus, the *payoff* graph for a portfolio can be modified via vertical shift by lending (purchasing a bond) or borrowing (issuing/shorting a bond) at the risk-free rate; however, the *profit* graph will be unaffected by the bond's appearance in the portfolio.

Payoff/profit for a stock/index. If, at time 0, we spend $S(0)e^{-\delta T}$ either to enter (a long position in) a prepaid forward agreement on an asset or to enter a (long) tailed position in that asset[3], we will have a payoff at expiration (time T) equal to $S(T)$. A short position produces a payoff of $-S(T)$. The profit is computed, as usual, by adjusting the payoff to reflect the time-T value of any time-0 costs or income:

Table 1.8: Cost, payoff, and profit for long and short prepaid forward positions in an investment asset

	Long position in S	Short position in S
Time-0 cost/income	cost of $S(0)e^{-\delta T}$	income of $S(0)e^{-\delta T}$
Time-T Payoff	$S(T)$	$-S(T)$
Time-T Profit	$S(T) - S(0)e^{(r-\delta)T}$	$-S(T) + S(0)e^{(r-\delta)T}$

What if a forward contract is used to acquire $S(T)$ rather than a prepaid forward contract? A forward contract has no cost/income at time 0, and at expiration the forward price $S(0)e^{(r-\delta)T}$ is exchanged for one share worth $S(T)$. So the payoff is $S(T) - S(0)e^{(r-\delta)T}$ for a long position in a forward contract and $S(0)e^{(r-\delta)T} - S(T)$ for the short position. The profit is equal to the payoff because no money changes hands until the delivery date of the forward contract. Note that the forward contract and the prepaid forward contract result in the same time-T profit.

[3]We follow the convention that all dividends are payable continuously unless otherwise noted. This simplifying assumption is actually somewhat realistic in the case of a stock index—the many stocks that make up the portfolio will pay various dividends at dates spread throughout the year, and our "δ" can be thought of as an appropriately weighted average of these dividends.

Exercises

Except where noted, the options in the following exercises are all European-style options, and there are no transaction costs.

Time Value of Money.

1. You will receive \$100 six months from now and \$150 six months later. Find the present (i.e. time-0) value of these cash flows if

   (a) the annual effective interest rate is 5%.

   (b) the continuously compounded interest rate is 5%.

Forward Contracts.

2. A stock currently sells for \$50/share. The continuously compounded risk-free rate is 4%. The stock continuously pays a dividend that is proportional to $S(t)$: $\delta = 3\%$.

   (a) You agree to pay an amount X six months from now in exchange for acquiring one share of the stock today. Find X. What is the name of this type of arrangement?

   (b) Calculate the amount of interest that you will pay.

3. A stock pays dividends continuously at a rate of 4%. The spot price for the stock is $S(0) = \$30$. Determine the present value of the dividend payments during $t \in [0, 1]$ corresponding to a single share of the stock.

   Hint: Find a combination of long and short positions involving outright purchase and a prepaid forward for one share such that the resulting cash flows replicate the stream of dividend payments.

4. A stock currently sells for \$50/share. The continuously compounded risk-free rate is 4%. The stock continuously pays a dividend that is proportional to $S(t)$: $\delta = 3\%$.

   (a) If you pay now to acquire the stock six months from now, what is the price? What type of contract is this?

   (b) You agree to pay an amount X six months from now to receive delivery of the asset at that time. Find X. What is the name of this type of contract?

5. Value of a forward contract. At time $t = 0$, you entered into a one-year forward contract ("Contract I") on a stock with delivery price \$40.10.

   The stock pays dividends continuously at a rate of 5%. The risk-free rate is 6%. At time $t = .25$, the stock's value is \$40.

   (a) At time $t = .25$, what is the delivery price for a 9-month forward contract ("Contract II") on the stock? (To clarify, delivery for Contract II is at $t = 1$.)

   (b) At time $t = .25$, what is the value of the one-year forward contract that you have been holding for three months (i.e., what is the value of "Contract I" at $t = .25$)?

6. Value of a forward.

   (a) Today, at time $t = 0$, a stock sells for \$25. The stock pays dividends at a continuous rate $\delta = .01$, and the risk-free rate is $r = .05$. You enter into an 18-month forward contract. What is the delivery price?

   (b) Later, at time $t = 1$ year, the stock's price is \$26.10. What is the delivery price for a 6-month forward on the stock, where the contract is signed at time $t = 1$?

   (c) At time $t = 1$, what is the value of the contract that you hold from part (a)?

7. Continuous dividends and tailed positions. Recall that the phrase "pays dividends continuously at annual rate 3%" means that, during a short time interval $[t_0, t_0 + dt]$ the shareholder receives (for each share owned) a dividend payment in the amount of $\delta \times S(t_0) \times dt$.

   (a) Consider a stock that pays dividends continuously at $\delta = 3\%$. Use $dt = 1/365$ to approximate the total amount of dividend payment over a three-day period $t \in [0, 3/365]$ if $S(0) = \$60$, $S(1/365) = \$63$, and $S(2/365) = 57$.

   (b) Suppose that, for $t = 0,\ 1/365,\ 2/365$, the dividend payment corresponding to $[t, t + 1/365]$ can be used to purchase additional shares at a price of $S(t)$, using the prices and stock from (a). Find the number of shares that can be purchased in this manner for the three-day period commencing at time 0.

8. A tailed position in an asset is used so that the number of shares grows according to the differential equation

$$dN(t) = \delta \cdot N(t)\,dt\,.$$

Verify that this differential equation, together with the decision to purchase $N(0) = e^{-\delta T}$ shares initially, implies that the number of shares at time T is $N(T) = 1$ share. (This completes the proof of Theorem 1.8.)

9. A stock pays dividends continuously at a rate of δ. Consider a prepaid forward contract and a forward contract, both written at time 0 and expiring at time $T > 0$.

   (a) Order the quantities $S(0)$, F^P, and F from least to greatest if $\delta > r > 0$.

   (b) Order the quantities $S(0)$, F^P, and F from least to greatest if $r > \delta > 0$.

10. Comparing contracts with different expiration dates. Let $0 < T_1 < T_2$, and for $i = 1,\ 2$ let $F(T_i)$ and $F^P(T_i)$ denote, respectively, the forward and prepaid forward prices for contracts written at time 0 with delivery date T_i.

    (a) If $\delta > 0$, how do the prepaid forward prices $F^P(T_1)$ and $F^P(T_2)$ compare?

    (b) If $r > \delta$, how do the forward prices $F(T_1)$ and $F(T_2)$ compare?

    (c) If $\delta > r$, how do the forward prices $F(T_1)$ and $F(T_2)$ compare?

11. Discrete dividends. A stock currently sells for \$50/share. The continuously compounded risk-free rate is 4%. The stock pays a single dividend of \$4 at time $t = .2$.

    (a) Find the 6-month prepaid forward price.

    (b) Find the 6-month forward price.

    (c) You enter an agreement to receive one share now and pay for it in 6 months. What will you pay?

12. Payoff for forward contracts. Consider a forward contract on a stock with delivery six months from now. The stock's current price is \$300, the stock does not pay dividends, and $r = 5\%$.

    (a) Compute the payoff for a long position in the forward contract if (i) $S(.5) = \$280$; (ii) if $S(.5) = \$320$.

(b) Repeat (a) for a short position in the forward contract.

13. Arbitraging a mispriced forward contract. The spot price for a nondividend paying stock is $S(0) = 50$. You are offered the opportunity to take a position in a forward contract that has forward price $F = \$51$ for delivery in six months. The risk-free rate is $r = 6\%$.

(a) Compute the "theoretical" forward price using the usual formula for F, and determine whether the offered \$51 forward price is higher or lower than it "should be".

(b) Consider the following strategy:

- Sell one share of stock outright at the \$50 spot price.
- Lend the \$50 income from shorting the stock at the risk-free rate (that is, buy a risk-free bond).
- Take a long position in the forward contract that has $F = \$51$.

This strategy has no net cost at time 0. Show that it produces arbitrage profit by evaluating the payoff for the strategy.

14. Arbitraging a mispriced forward contract. The spot price for a nondividend paying stock is $S(0) = 50$. You are offered the opportunity to take a position in a forward contract that has forward price $F = \$52$ for delivery in six months. The risk-free rate is $r = 6\%$.

(a) Compute the "theoretical" forward price using the usual formula for F, and determine whether the offered \$52 forward price is higher or lower than it "should be".

(b) Determine a strategy that will yield arbitrage profit. The strategy should involve purchasing one share, borrowing funds (equivalently, taking a short position in a risk-free bond), and taking the appropriate long or short position in a forward contract on one share.

(c) Demonstrate that your strategy has no time-0 net cost, and demonstrate that it produces arbitrage profit by computing the payoff.

15. Arbitrage and transaction costs. Consider the situation of Example 1.14, except assume that continuously compounded rate (let's call it r^*) at which the investor can borrow funds is not known. Find the inequality that gives all possible values of r^* for which the following strategy results in arbitrage profit:

- Take a short position in the forward contract. (This incurs a \$1 transaction fee at time 0).

- Buy one share at time 0 (this also incurs a \$1 transaction fee).
- Borrow enough money (at time 0) at r^* to cover both the purchase of the share and the transaction costs.

Futures contracts.

16. Futures contracts and margin accounts. Review the context and data of Example 1.15 in Section 1.3.2.

    (a) On January 1, the futures price (for delivery of 10 shares of the stock index on July 1) was \$2011. At the closing bell on January 1, what was the purchase price for 10 shares? (Assume there are 180 days remaining until expiration; that's 180/365 of a year.)

    (b) At the end of the day on January 4, the futures price (for delivery of 10 shares on July 1) was \$2006.11 and the margin account's balance was \$2007.10. At the end of the day on January 5, the futures price (for 10 shares) was \$2005.11. Determine the new balance of the margin account.